

The Dynamic Welfare Cost of Inflation Tax and Consumption Tax in a Cash-in-Advance Model*

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September 2008

Abstract

Using a framework that combines monetary policy with public finance, this paper studies the welfare costs between a seignorage tax and a consumption tax. One of these two taxes is used to finance an exogenous stream of public spending in a fashion to balance the government budget. The welfare cost of a consumption tax is lower if only equilibrium paths in a steady state are compared. This paper compares equilibrium paths along both transitional dynamics and steady state. In an optimal growth model, the welfare cost of a consumption tax is always higher on the condition of lower relative prices of consumption (to investment) in a seignorage tax than in a consumption tax in early periods of tax changes. In permanent taxes, the condition is that a fraction of investment is cash constrained. In temporary taxes, there is no such a requirement of cash constraints on investment. Temporary taxes have intertemporal substitution effects away from current toward future consumption while a temporary consumption tax marks up and leads to a higher relative price of consumption in early periods. This welfare ranking order holds true not only in an optimal growth model but also in an endogenous growth model. Given the result of a lower welfare cost of a seignorage tax, our result lends supports to Phelps (1973) and is in favor of an optimal inflation tax higher than the Friedman (1969) rule.

Keywords: growth model, cash-in-advance, seignorage tax, consumption tax, welfare cost.

JEL classification: E13, E52, O23.

* Part of the research was conducted when the authors visited the Department of Economics, Washington University in St. Louis. They thank the hospitality offered by the Department, especially Ping Wang, the department chair.

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1 Introduction

Monetary economists have devoted considerable effort to measuring the welfare cost of inflation. The traditional approach treats real balances as a consumption good and inflation as a tax on real balances. Although some studies found a large welfare cost of inflation and others uncovered a small welfare cost of inflation, their findings established a negative relationship between inflation and welfare.¹ In this body of research, the optimal inflation tax is such that the nominal rate of interest calls for deflation at the rate of time preference, or a zero nominal interest rate, known as the Friedman rule (Friedman, 1969).

Alternatively, there is a line of research that studies monetary policy in a public finance approach. This strand of research was started by Phelps (1973). In an environment where a lump-sum tax is not available, Phelps (1973) argued against the Friedman rule and showed that all goods, including real balances, would be taxed in a Ramsey-like fashion. In dynamic general equilibrium models, the issue was re-examined by Chari, et al. (1991) and Braun (1994) in terms of the welfare cost between a seignorage tax and an income tax. These two sets of authors employed a Lucas and Stokey (1983) model without capital and with one cash goods and one credit goods. Under a class of preference, the results in Chari, et al. (1991) were supportive of the Friedman rule of zero nominal interest rate, while under a more general specification of preference, the conclusion in Braun (1994) was in favor of a seignorage tax above the Friedman rule.²

Later, Palivos and Yip (1995) and Holman and Neanidis (2006) followed the public finance approach used by Chari, et al. (1991) and Braun (1994). These two papers investigated endogenous growth models with either a cash-in-advance constraint (Palivos and Yip, 1995) or a liquidity constraint with currency substitution (Holman and Neanidis, 2006). In these two works, consumption and a fraction of investment are cash or liquidity constrained. These two studies compared the welfare cost between a seignorage tax and an income tax as alternative ways of financing exogenous public spending along a balance growth path, as do Chari, et al. (1991) and Braun (1994).³ They found that in the long run, the welfare cost of a seignorage tax is lower than

¹ Examples include Fischer (1981), Cooley and Hansen (1989), Kimbrough (1986), Prescott (1987), Cole and Stockman (1992), Gillman (1993), Gomme (1993), Ireland (1994, 2007), Dotsey and Ireland (1996), Aiyagari, et al. (1998), Wu and Zhang (2000) and Erosa and Ventura (2002). See Lucas (2000) for a survey.

² While Chari, et al. (1991) required the preference to be homogenous in the two consumption goods and weakly separable in leisure activities, Braun (1994) assumed a more general preference that includes the one used in Chari, et al. (1991) as a special case.

³ In each of the above-mentioned studies public spending was wasted save for Holman and Neanidis (2006) wherein public spending enhanced the production function as in Barro (1990).

that of an income tax on the condition that a sufficiently large fraction of investment is with a cash or a liquidity constraint.⁴ Their results are in supportive of Braun (1994) and against Chari, et al. (1991) and suggested an optimal inflation tax higher than the Friedman rule. All the above papers in dynamic general equilibrium models compared a seignorage tax with an income tax, a tax that has known to be dynamically inefficient in the long run since Judd (1985) and Chamley (1986).

In this paper, we extend the body of research in the public finance approach and compare the welfare cost between a seignorage tax and a consumption tax. A comparison of a seignorage tax with a consumption tax is valuable and important for two reasons. First, in an economy with money as a medium of exchange, consumption is usually more cash constrained than other market activities. A seignorage tax amounts to taxing consumption at a higher rate than other market activities. Such a feature makes a seignorage tax close to a consumption tax and as a result and as such is useful in comparing these two types of taxes. Second, a consumption tax is dynamically neutral and is less distortionary than an income tax. Because of this reason, recent tax reforms in advanced economies connote a trend toward the reliance on consumption taxes in place of income taxes. A comparison of a consumption tax with a seignorage tax may shed light on such a tax reform and reveal whether or not the direction of reform is appropriate in terms of the welfare criterion.

This paper studies an otherwise standard optimal growth model with a cash-in-advance (henceforth, CIA) constraint. There is an exogenous stream of waste public spending financed either by a seignorage tax or by a consumption tax. If we only consider equilibrium paths along a steady state, as in Palivos and Yip (1995) and Holman and Neanidis (2006), the welfare cost of a seignorage tax is always higher than that of a consumption tax when a fraction of investment is cash constrained. This result arises because a consumption tax is neutral while a seignorage tax hurts capital formation in the long run; therefore the level of consumption and thus welfare is lower in a seignorage tax in the long run.

However, if we take into account equilibrium paths along transitional dynamics, the welfare ranking order is reversed if a seignorage tax increases the price of consumption (to investment) is less than what a consumption tax does in early periods. Under such a condition, a consumption

⁴ In an endogenous growth model with a loan constraint in a Diamond and Dybvig (1983) framework, Espinosa-Vega and Yip (2002) also compared the welfare cost of a seignorage tax with that of an income tax. They failed to support an optimal inflation tax higher than the Friedman rule as they could not meet the condition that the binding loan constraint for welfare-maximization be smaller than the binding loan constraint for inflation-minimization, according to their proposition 7.

tax has lower consumption than a seignorage tax in early periods whose adverse effect on welfare dominates the effect of higher consumption in a consumption tax in a steady state. In the case of a permanent tax, there is a difference in relative prices if there is a fraction of investment with a cash constraint. As investment requires a fraction of real balances, a seignorage tax raises the relative price of consumption less than what a consumption tax does. In the case of a temporary tax, a cash constraint on investment is not required. Both temporary taxes have an intertemporal effect away from current toward future consumption. Moreover, a temporary consumption tax marks up the price of current consumption relative to investment as opposed to a temporary seignorage tax which, other things being equal, increases the general price for both consumption and investment without changing the relative price of current consumption to current investment. As a result, even when investment is not cash constrained, consumption is reduced more in a temporary consumption tax in early periods of tax changes whose adverse effect on welfare dominates that of higher consumption in a consumption tax in latter periods of tax changes. Therefore, the welfare cost of a consumption tax is higher than the welfare cost of a seignorage tax. These results hold true not only in an optimal growth model but also in an endogenous growth model. Thus, even in comparisons with a consumption tax that is neutral to long-run capital formation, a seignorage tax has a lower welfare cost. As a result, our findings are in favor of an inflation tax that is higher than the Friedman rule, as found in Braun (1994), Palivos and Yip (1995) and Holman and Neanidis (2006).⁵

In a recent paper by Ho, et al. (2007), a comparison of the welfare cost between a seignorage tax and a consumption tax has been made. Similar to our study, they found that under some conditions, the welfare cost in a consumption tax is higher than that of a seignorage tax. Our study is differentiated from their work in the following important aspects. First, they use a money-in-utility model while we use a cash-in-advance model. As a result, while money in their model has a store of value, money in our model only has only the transaction role, which makes money closer and more compatible to consumption. Second, they obtain a higher welfare cost in a consumption tax than in a seignorage tax based on the condition of a sufficiently large externality in production, an assumption that is challenged by empirical studies (e.g., Burnside, 1996; Basu and Fernald, 1997). Finally and more importantly, their paper only compares the welfare effect in

⁵ Recently, Waller (2008) compared taxes on consumption of the old and inflation taxes and found that the Friedman rule is not optimal in the symmetric equilibrium. Waller (2008) used a search-theoretic model in which only a fraction of agents hold cash. In particular, an agent must obtain real balances before entering the old age. The taxation of consumption at the old age is this like a tax on monetary wealth.

a steady state, as do Palivos and Yip (1995) and Holman and Neanidis (2006). Our comparisons are made for the whole equilibrium path along transitional dynamics as well as in a steady state. Our approach proves to be important as the effect of a policy change needs to be evaluated not only in a long run but also in a short run along transitional dynamic paths.

The remainder of this paper is organized as follows. Section 2 sets up a basic one-sector, dynamic general equilibrium growth model with a cash-in-advance constraint and studies the optimization conditions. Section 3 examines the case of an optimal growth model, whereas section 4 presents an endogenous growth model. We study the welfare costs of alternative ways of finance in these two models. Finally, some concluding remarks are offered in Section 5.

2 The Basic Model

2.1 Environment

The model is based on Stockman (1981) and Rebelo (1991). At any point of time, a representative agent's lifetime utility is

$$U = \int_0^{\infty} u(c) e^{-\rho t} dt, \quad (1)$$

where c is consumption and $\rho > 0$ is the time preference rate. The felicity $u(c)$ has standard properties and is increasing and concave in c .

The representative individual's budget constraint is

$$\dot{k} + \dot{m} = y - (1 + \tau_c)c - \pi m - \delta k, \quad (2)$$

where y is the output per capita, k capital per capita, m real money holdings per capita, τ_c consumption tax rate, π inflation rate, and δ depreciation rate of capital.⁶ Real disposable income after consumption is held either in terms of capital or real balances.

Output per capita y is produced according to technology $f(k)$ which has a standard positive marginal product of capital per capita; i.e., $f'(k) > 0$. We will consider two models concerning the way the marginal product of capital changes in capital. We assume $f''(k) < 0$ in an optimal growth model, while $f''(k) = 0$ in an endogenous growth model.

If we denote I as the gross investment per capita, net investment is as follows.

$$\dot{k} = I - \delta k. \quad (3)$$

Both consumption and investment may be subject to CIA constraints. We consider

⁶ To save space we do not consider income taxes. In endogenous growth models with a cash-in-advance or a liquidity constraint, Palivos and Yip (1995) and Holman and Neanidis (2006) compared income taxes and inflation taxes and obtained findings in favor of an inflation tax that is higher than the Friedman rule.

$$c(1 + \tau_c) + \varphi I \leq m, \quad 0 \leq \varphi \leq 1. \quad (4)$$

Because taxes are paid in cash, consumption taxes $\tau_c c$ are a part of the CIA constraint. The above CIA constraint is the one utilized in Wang and Yip (1992) and Palivos and Yip (1995) and is general that includes many special cases. If $\varphi=1$, consumption and investment are both equally cash constrained. Such a constraint is employed by Stockman (1981) and Abel (1985). If $\varphi=0$, only consumption is liquidity constrained as assumed in Clower (1967) and Lucas (1980).⁷ While our ranking about welfare comparisons below requires $\varphi>0$ in the event when a permanent source of revenues is necessary, the ranking comparison holds under $\varphi=0$ in the event when a temporary source of revenues is required. This holds true both in a neoclassical growth model and an endogenous growth model.

To complete the model, we specify the government behavior. We assume that at any point in time the government may finance an exogenously given expenditure sequence $\{G(t)\}$ that is not valued by private agents. As in the line of research in public finance approach, we abstract the effects of government expenditure and assume that the government expenditure affects neither the production nor preferences. This assumption isolates the distortions generated by the government expenditure. Money supply and consumption taxes are thus the two sources of distortions.

Following existing studies, we assume that the government spending is a fraction of total output: $G(t)=\beta y(t)$, where $0 \leq \beta < 1$.⁸ To finance its expenditures, the government relies on two sources of revenue: a seignorage tax (or the additional printing of money) and a consumption tax. The money supply is initially predetermined. Let μ be the growth rate of nominal money. The government budget constraint is given by

$$\mu m + \tau_c c = G = \beta f(k). \quad (5)$$

2.2 Optimization Conditions

The representative agent's problem is to maximize (1), subject to (2)-(4), taking as given the monetary growth rate, the tax rate, initial capital and initial nominal money holdings. Let $\lambda_k > 0$ and $\lambda_m > 0$ be the co-state variables associated with capital and real monetary holdings, respectively, and $\xi > 0$ be the Lagrange multiplier of the CIA constraint. The necessary conditions are

$$u'(c) = (1 + \tau_c)[\lambda_m + \xi], \quad (6a)$$

⁷ In this case consumption is a cash good and investment is a credit good. As investment is produced from the same sector as the consumption good but is not used for consumption in our model, the nature of our credit good is different from that in Lucas and Stokey (1983) and Chari, et al (1991).

⁸ For example, see Cooley and Hansen (1991), Palivos and Yip (1995), Lucas (2000) and Ho, et al. (2007).

$$\lambda_k = \lambda_m + \xi\varphi, \quad (6b)$$

$$\dot{\lambda}_k = (\rho + \delta)\lambda_k - f'(k)\lambda_m, \quad (6c)$$

$$\dot{\lambda}_m = \rho\lambda_m - [-\lambda_m\pi + \xi], \quad (6d)$$

and the transversality conditions $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_{kt} k_t = 0$ and $\lim_{t \rightarrow \infty} e^{-\rho t} \lambda_{mt} m_t = 0$.

Under these conditions, (6a) equalizes the marginal utility of consumption to the marginal cost of consumption. The marginal cost of consumption includes not only the shadow price of real balances but also the shadow price of the CIA constraint on consumption. In (6b) optimal investment requires no arbitrage between capital and real balances: the shadow price of capital equals the shadow prices of real balances and the CIA constraint on investment. Finally, conditions (6c) and (6d) are the intertemporal no-arbitrage conditions which control how each of the two co-state variables changes over time.

2.3 Equilibrium Conditions

In equilibrium, the money and the goods markets are both clear. The goods market clearing condition is $y = c + \dot{k} + \delta k + G$ which, using (5), becomes

$$\dot{k} = (1 - \beta)f(k) - \delta k - c. \quad (7a)$$

The money market clearing condition is

$$\dot{m} = (\mu - \pi)m. \quad (7b)$$

Perfect-foresight equilibrium is a time path $\{c, m, k, \lambda_k, \lambda_m, \xi, \pi\}$ that satisfies the agent's optimization, (6a)-(6d), the money and the goods market equilibrium, (7a)-(7b), and the binding CIA constraint (4).⁹ We are ready to analyze the equilibrium and the welfare costs of alternative public finance.

We are interested in the effects of alternative public financing: an exogenous change in the share of public expenditure β financed either by a change in the consumption tax rate τ_c or a change in the rate of monetary growth μ in order to balance the government budget. We consider two types of policy changes. We first analyze the effect of a permanent increase in the share of public spending, followed by a temporary increase in the share of public spending. To simplify the analysis, in what follows we assume that initially there is no government expenditure and thus,

⁹ Following Lucas (1980) and Wang and Yip (1992), we only consider the case when the CIA constraint is binding in equilibrium.

both the tax rate and the monetary growth rate are zero; i.e., $\beta=\tau_c=\mu=0$.

We start with an optimal growth model, followed by an endogenous growth.

3 Optimal Growth Model

Suppose that the economy is in the world of an optimal growth model. To determine the equilibrium, first, if we substitute ξ in (6b) into (6a), we obtain

$$\lambda_k = \frac{\varphi u'(c)}{1+\tau_c} + (1-\varphi)\lambda_m \equiv \lambda_k(c, \lambda_m). \quad (8a)$$

Next, differentiating (6a) with respect to time, with the use of (6c) and (8a), yields

$$\dot{c} = \frac{(1+\tau_c)}{-u''(c)\varphi} [f'(k)\lambda_m - (\varphi-1)\dot{\lambda}_m - (\rho+\delta)\lambda_k(c, \lambda_m)], \quad (8b)$$

which is the Keynes-Ramsey condition.

Moreover, (3) and (7a) indicate $(1-\beta)f(k)=c+I$, and the CIA constraint in (4) suggests $m=(1+\tau_c-\varphi)c+\varphi(1-\beta)f(k)$. If we differentiate this relationship and use (7b), we attain

$$\pi = \mu - \frac{(1+\tau_c-\varphi)\dot{c}+\varphi(1-\beta)f'(k)\dot{k}}{(1+\tau_c-\varphi)c+\varphi(1-\beta)f(k)}.$$

By substituting \dot{c} in (8b) and \dot{k} in (7a), along with $\dot{\lambda}_k$ in (6c) and $\dot{\lambda}_m$ in (6d), the above expression leads to the following relationship

$$\pi = \pi(c, k, \lambda_m; \beta, \tau_c, \mu). \quad (8c)$$

Finally, substituting ξ in (6b) into (6d), together with (8a) and (8c), yields

$$\dot{\lambda}_m = \lambda_m \left[\rho + \frac{1}{\varphi} + \pi(c, k, \lambda_m; \beta, \tau_c, \mu) \right] - \frac{\lambda_k(c, \lambda_m)}{\varphi}. \quad (8d)$$

Thus, the equilibrium system is simplified to three equations, (7a), (8b) and (8d). These equations determine the equilibrium paths of c , k and λ_m . The equilibrium paths of λ_k , ξ , π and m are in turn determined by (8a), (6b), (8c) and (4), respectively.

In a steady state, $\dot{c} = \dot{k} = \dot{\lambda}_m = \dot{m} = 0$. A steady state may be determined in a recursive manner as follows. According to $\dot{m} = 0$ in (7b), inflation is $\pi^* = \mu$ in a steady state.¹⁰ Next, in a steady state, the locus $\dot{c} = 0$ in (8b) is vertical in a (k, c) plane, while the locus $\dot{k} = 0$ in (7a) is positively and then negatively sloping in a (k, c) plane. See Figure 1.

[Insert Figure 1 about here]

We thus determine a unique pair of (k^*, c^*) in a steady state as follows.

$$f'(k^*) = (\rho + \delta)[1 + \varphi(\rho + \mu)], \quad (9a)$$

¹⁰ An asterisk is used to denote values in a steady state.

$$c^* = (1 - \beta)f(k^*) - \delta k^*. \quad (9b)$$

Moreover, using $\dot{\lambda}_m = 0$ we obtain

$$\lambda_m^* = \frac{u'(c^*)}{(1 + \tau_c)(1 + \rho + \mu)}, \quad (9c)$$

and using the CIA constraint, we obtain

$$m^* = (1 + \tau_c)c^* + \phi \delta k^*. \quad (9d)$$

Thus, we have completely determined the unique steady state.

Finally, along the equilibrium path in a steady state, the level of utility is

$$U^* = \frac{u'(c^*)}{\rho}. \quad (9e)$$

3.1 Steady State Effect

To characterize the steady-state effect of alternative public finance, we consider a permanent increase in the share of government spending in total output, $d\beta > 0$.¹¹ The following constraint must be met in order to balance government budget. In deriving the following equation, it is evaluated at the initial conditions $\beta = \mu = \tau_c = 0$.

$$d\beta = \frac{1}{f(k^*)}(m d\mu + c d\tau_c).$$

First, if public spending is financed only by consumption taxes, the consumption tax rate is

$$d\tau_c = \frac{f(k^*)}{c^*} d\beta. \quad (10a)$$

Alternatively, if a higher growth rate of monetary supply is used, then the growth rate of monetary supply is

$$d\mu = \frac{f(k^*)}{m^*} d\beta. \quad (10b)$$

It is clear from (9a) that capital is not affected by consumption taxes in the long run. Yet, using the goods market equilibrium condition in (7a), consumption in the long run is adversely affected by

$$dc^* \Big|_{d\tau_c > 0} = -f(k^*) d\beta < 0,$$

which in turn reduces the level of lifetime utility in a steady state by

$$dU^* \Big|_{d\tau_c > 0} = \frac{u'(c^*)}{\rho} dc^* = -\frac{u'(c^*)}{\rho} f(k^*) d\beta < 0. \quad (11a)$$

¹¹. In Appendix 1, we have shown that the steady state is a saddle and thus for a given initial state, there is a unique equilibrium path toward the steady state.

In the case of seignorage taxes, capital is reduced by $dk^* = \varphi \frac{(\rho+\delta)}{f''(k^*)} d\mu = \varphi \frac{(\rho+\delta)}{f''(k^*)} \frac{f(k^*)}{m^*} d\beta < 0$ if $\varphi > 0$, but is not affected in the long run if $\varphi = 0$. Using (7a) and (10b), consumption in the long run is reduced by

$$dc^* \Big|_{d\mu > 0} = -m^* d\mu + [(1-\beta)f'(k^*) - \delta]dk^* = -f(k^*)d\beta + [(1-\beta)f'(k^*) - \delta]dk^* < 0,$$

As a result, the welfare in a steady state is then reduced by

$$dU^* \Big|_{d\mu > 0} = -\frac{u'(c^*)}{\rho} f(k^*)d\beta + \frac{u'(c^*)}{\rho} [(1-\beta)f'(k^*) - \delta]dk^* < 0. \quad (11b)$$

The righthand side of (11b) is reduced to (11a) if only consumption is cash constrained ($\varphi = 0$) and thus $dk^* = 0$. When $\varphi > 0$, (11b) is smaller than (11a). Thus, we obtain

Proposition 1 *In an optimal growth model when investment is cash constrained, in terms of the welfare criterion a consumption tax is always less costly than a seignorage tax in the long run.*

The above results are reasonable. Because public spending has a crowding-out effect, no matter which way of finance there is a negative effect on consumption in the long run, thereby reducing the representative agent's welfare in the long run. Moreover, a seignorage tax reduces capital formation in steady state when investment is cash constrained while a consumption tax does not. As a result, in terms of welfare, a permanent consumption tax is better than a permanent seignorage tax in the long run.

Our above result is similar to the welfare ranking order in the money-in-utility model by Ho, et al. (2007) when productive externalities are not sufficiently large. This similarity of results in the long run is reminiscent of Wang and Yip (1992) where they showed a long-run qualitative equivalence between a CIA model when consumption and investment are both cash constrained and a money-in-utility model when leisure is complement to consumption and real balances.

Alternative ways of public finance exert effects on allocation not only in the long run but also in the transitional path before a steady state is reached. In assessing the relative effects on welfare, it is thus important to take account of all the effects before and along a steady state. As we will see, the welfare ranking order is easily reversed once the effects along transitional dynamics are taken into account.

3.2 Dynamic Effects

We now analyze the welfare ranking order when effects along transitional dynamics are taken into account. As the system in our equilibrium conditions is 3 dimensioned, it is not possible to

provide an analytical, even a graphical, illustration of the dynamic effects. Therefore, we offer a quantitative analysis.

To conduct a quantitative analysis, we assume a felicity with a constant intertemporal elasticity of substitutability (henceforth, IES): $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, where $\sigma > 0$ is the inverse of the intertemporal elasticity of substitution. A felicity with a constant IES is consistent with a balanced growth path. We assume a Cobb-Douglas technology: $f(k) = Ak^\alpha$, where $0 < \alpha < 1$ is the share of capital in production and $A > 0$ is the coefficient of productivity. We calibrate the model in a steady state in order to reproduce some key features representative of the U.S. economy, using yearly data.

Following Cooley (1995), the annual rate of capital depreciation is chosen at $\delta = 5\%$, the rate of time preference at $\rho = 4\%$, and the value of the reciprocal of the IES in consumption at $\sigma = 2.5$ in our benchmark model. According to Wang and Yip (1992), we set the liquidity constraint parameter on investment at $\varphi = 0.5$. In the theoretical model we assume the initial size of government expenditure is zero. To be consistent, we set $\beta = \mu = \tau_c = 0$ in calibration. When we change β in quantitative analysis later, both μ and τ_c are endogenously determined in a way to balance the government budget. Finally, we normalize $A = 1$, and then calibrate the share of capital in production in consonance with the annual capital-output ratio at 3.32 (Cooley, 1995, p.21). We obtain $\alpha = 0.31$. Under these parameter values, we obtain a unique equilibrium in a steady state: $k^* = 5.834$, $c^* = 1.4359$, $y^* = 1.7276$, $\lambda_m^* = 0.3892$ and $U^* = 6.9805$.

To quantify the effects on welfare both along transitional dynamics and steady state, we take a linear Taylor's expansion of the equilibrium system (7a), (8b) and (8d) in the neighborhood of the unique steady state and obtain a Jacobean matrix. We find a negative eigenvalue (stable root) and two positive eigenvalues associated with the Jacobean matrix which guarantee a unique equilibrium path toward the steady state. The values of $c(t)$, $k(t)$, $\lambda_m(t)$ in the unique equilibrium path are then each represented by the sum of their own new steady state c^* , k^* , λ_m^* and a product of three components: a coefficient, an exponential to the power of the negative eigenvalue times the time, and the corresponding eigenvector of the negative eigenvalue. The coefficient is determined by boundary conditions. Two kinds of experiments are conducted. We start by the effects of a permanent tax change, followed by the effect of a temporary tax change. In Appendix 1 and 2 we offer details concerning the determination of the equilibrium paths of the endogenous variables under a permanent and a temporary tax change, respectively.

3.2.1 A permanent policy

Suppose that the government increases the share of public expenditure from 0% to 10% of total output once and for all. When the spending is financed by a seignorage tax, a higher rate of monetary growth rate at each point in time is required in order to balance the government budget. Alternatively, in a consumption tax, it is necessary to levy a consumption tax rate in order to balance the government budget. The corresponding monetary growth rate and the consumption tax rate are reported in panel A of Figure 2. The effects on the equilibrium paths of capital stock and consumption are also reported in panel A of Figure 2.

[Insert Figure 2 about here]

While a permanent consumption tax has a neutral effect on capital, a permanent seignorage tax has an adverse effect on capital. Both types of taxes generate harmful effects on consumption, with a stronger adverse effect under a seignorage tax in later periods near a steady state. However, the adverse effect under a consumption tax is stronger in earlier periods. The net effect on the welfare depends on whether the effect in early periods or the effect in later periods dominates. Under our calibrated discount rate of $\rho=0.04$, the discounted utility under a consumption tax is lower than the discounted utility under a seignorage tax. See panel A in Table 1. As the discounted lifetime utility depends on the discount rate, one may wonder if other discounted rates may lead to a reverse welfare ranking order. We have experimented with the effects under many rates of time preferences from $\rho=0.001$ to $\rho=1$. With each ρ we recalculate the model under $\beta=\tau_c=\mu=0$ in order to attain the equilibrium. We then increase β by 10% and calculate the welfare change under each way of finance. As our results show, regardless of the value of the rate of time preference, the welfare under a consumption tax is always lower than that under a seignorage tax. Therefore, the welfare cost of a consumption tax is higher than the welfare cost of a seignorage tax.

[Insert Table 1 about here]

In the special case when investment is not cash constrained ($\varphi=0$), equilibrium paths are illustrated in panel B of Figure 2.¹² There, a permanent seignorage tax and a permanent consumption tax both have the same negative effect on consumption with neutral effects on capital accumulation. These two taxes raise same relative prices of consumption when $\varphi=0$. However, when investment is slightly cash constrained, $\varphi>0$, equilibrium paths behave somewhat between panel A and panel B of Figure 2. A permanent seignorage tax now raises the relative price of

¹² With each φ we recalculate the model under $\beta=\tau_c=\mu=0$ in order to attain the equilibrium. We then increase β by 10% and calculate the welfare change under each way of finance.

consumption less than in a permanent consumption tax because investment is constrained by tighter real balances. The level of consumption under a seignorage tax is higher than under a consumption tax in early periods and this effect dominates the effect in the long run when, due to an adverse effect on investment in early periods, a seignorage tax has lower capital and thus consumption in the long run. In our numerical exercises, when φ is reduced, the welfare cost difference is smaller but the welfare cost of a consumption tax is always higher than that of a seignorage tax even if φ is as small as 1% (panel A in Table 1).

Our results thus indicate that under all plausible discount rates, as long as there is a small fraction of investment that is cash constrained, the welfare cost under a consumption tax is higher than the welfare cost under a seignorage tax. Evidence indicates that firms are also cash constrained.¹³ Thus, the welfare cost is higher in a consumption tax than in a seignorage tax.

3.2.2 A Temporary policy

While some types of public spending are permanent, other kinds may last for finite periods and thus a temporary source of revenues is required. We now study a temporary tax change.

Suppose that the government increases the share of public expenditure from 0% to 10% of total output now and the spending lasts for 10 periods.¹⁴ The expenditure is financed by either temporary seignorage taxes or temporary consumption taxes. We choose a tax rate and a seignorage rate in order to balance the government budget at each point in time (panel A of Figure 3). The effects on equilibrium paths are reported in panel A of Figure 3.

[Insert Figure 3 about here]

Under a temporary consumption tax, there is an intertemporal substitution effect away from current toward future consumption. This exerts an adverse effect on capital formation, different from a permanent consumption tax in Figure 2. As a result, the level of consumption in early periods under a temporary consumption tax in Figure 3 is higher than that under a permanent consumption tax in Figure 2. On the other hand, due to an intertemporal substitution effect away from current toward future investment, a temporary seignorage tax reduces capital in Figure 3

¹³ Using data in the U.S, Bates, et al (2006) documented that firms seem to hold so much more cash in the past 20 years after 1980 with the average ratio of cash to total assets more than doubled, from 0.1038 in 1980 to 0.2304 in 2003. The evidence indicates that firms are also cash constrained

¹⁴ We may also consider an experiment of a one-period increase in public spending. We choose to do an exercise of a ten-period increase because the effects in the diagrams are more visible. The results are the same regardless of doing a one-period or a ten-period temporary increase.

much more than that under a permanent seignorage tax in Figure 2. As a result, the level of consumption under a temporary seignorage tax is reduced much less than that under a permanent seignorage tax.

Although the level of capital is lowered under a temporary consumption tax, over time its level remains higher than that under a temporary seignorage tax, as in the cases of permanent tax changes. Also, as in the case of permanent tax changes indicated in Figure 2, the level of consumption is lower under a temporary consumption tax in early periods while it is lower under a temporary seignorage tax in later periods. As a result, the relative discounted lifetime utility under alternative taxes may depend on the rate of time preference. In our numerical exercises using all rates of time preference that assure the existence of a steady state, the welfare under a temporary consumption tax is always lower than that under a temporary seignorage tax, a result consistent with permanent tax changes. See panel B, Table 1.

As φ is smaller, the results are unchanged. This result holds true even if $\varphi=0$ and investment is not cash constrained. See the equilibrium paths of consumption and capital under $\varphi=0$ in Panel B of Figure 3.¹⁵ Equilibrium paths in a temporary seignorage tax are now close to those in a temporary consumption tax. The welfare cost difference between the two taxes becomes smaller as φ is smaller, but even under $\varphi=0$, the welfare in a consumption tax is always lower than that in a seignorage tax as seen in panel B, Table 1.

Cash constraints on investment are not required in temporary tax changes for reasons as follows. Under both temporary seignorage and consumption taxes, there is an intertemporal substitution effect away from current toward future consumption. This effect tends to reduce current consumption under both taxes. Investment is also temporarily reduced because of temporarily lower disposable income in both taxes. Moreover, a temporary consumption tax marks up the price of current consumption relative to current investment, as opposed to a temporary seignorage tax which, other things being equal, increases the general price for both consumption and investment without changing the relative price of current consumption to current investment. As a result, even when investment is not cash constrained, consumption is reduced more in a temporary consumption tax in early periods of tax changes. Although the level of consumption in a seignorage tax is lower than that in a consumption tax in later periods of tax changes, no matter what feasible discount rates we use, the welfare loss in early periods always

¹⁵ While the charts in panel A, Figure 3, report equilibrium paths for a long period, the charts in panel B only report a part of equilibrium paths in the first 20 or 10 periods in order to visibly see the differences between the two paths in a temporary consumption tax and in a temporary seignorage tax.

dominates that in latter period, thereby making the welfare cost in a consumption tax higher than in a seignorage tax.

To summarize our results,

Proposition 2 *In an optimal growth model, when equilibrium paths along both transitions and steady state are taken into account,*

- (i) *a permanent consumption tax always has a higher welfare cost than a permanent seignorage tax if investment is cash constrained;*
- (ii) *a temporary consumption tax always has a higher welfare cost than a temporary seignorage tax.*

4 Endogenous Growth Model

This section turns our attention to the world of an endogenous growth model. The economic structure in this environment is the same as that of an optimal growth model in Section 3 except that now the marginal product of capital does not change in capital; i.e., $f'(k)=0$. To simplify the analysis we maintain a similar form of production to that in Section 3: $f(k)=Ak^\alpha$, where $\alpha=1$ so $f'(k)=0$. We continue to use the same form of felicity with a constant IES as in Section 3. In an endogenous growth model, equilibrium quantities and prices may increase without bounds and thus the equilibrium conditions may not be stationary. In order to analyze the model, it is necessary to transform these equilibrium conditions into conditions in terms of a stationary system. We briefly describe the transformation below.

Denote $z=c/k$, the ratio of consumption to capital, and $p=\lambda_k/\lambda_m$, the ratio of the price of capital to the price of real balances. Then, (7a) becomes $\dot{k}/k=(1-\beta)A-\delta-z$. Differentiating (6a) with respect to time with the use of (6b)-(6d), yields

$$\frac{\dot{c}}{c} = \frac{1}{\sigma[(1-p)/\varphi-1]} \left[\left(1 - \frac{1}{\varphi}\right) \frac{\dot{\lambda}_m}{\lambda_m} + \frac{1}{\varphi} p \frac{\dot{\lambda}_k}{\lambda_k} \right].$$

If we substitute $\dot{\lambda}_k/\lambda_k$ in (6c) and $\dot{\lambda}_m/\lambda_m$ into (6d), the above expression becomes

$$\frac{\dot{c}}{c} = \frac{1}{\sigma[(p-1)/\varphi+1]} \left\{ \frac{1}{\varphi} p \left[\frac{A}{p} - (\rho + \delta) \right] + \left(1 - \frac{1}{\varphi}\right) \left[\frac{p-1}{\varphi} - (\pi + \rho) \right] \right\}. \quad (12a)$$

As in Section 3, we focus on equilibrium when the CIA constraint is binding. A binding CIA constraint indicates $\xi > 0$, while (6b) suggests $p=1+\varphi\xi/\lambda_m \geq 1$. Note that a non-degenerated growth rate in (7a) requires

$$z \leq (1-\beta)A-\delta.$$

As (3) and (7a) lead to $(1-\beta)Ak=c+I$, a binding CIA constraint suggests

$$m=(1+\tau_c-\varphi)c+\varphi(1-\beta)Ak. \quad (12b)$$

If we differentiate this equation and use (7b), we attain

$$\pi = \mu - \frac{1}{(1+\tau_c)z+\varphi[(1-\beta)A-z]}[(1+\tau_c-\varphi)z\frac{\dot{c}}{c} + \varphi(1-\beta)A\frac{\dot{k}}{k}].$$

If we use \dot{c}/c in (12a) and \dot{k}/k in (7a), we may rewrite the above expression as

$$\pi = \pi(z, p; \mu, \tau_c, \beta), \quad (12c)$$

where in Appendix 4, we have shown $\pi_z > 0$, $\pi_\mu > 0$, $\pi_{\tau_c} = 0$ and $\pi_\beta > 0$, and π_p is ambiguous.

Finally, if we use (6c), (6d), (7a) and (12a), together with (12c), we obtain

$$\frac{\dot{p}}{p} = \delta + \frac{p-1}{\varphi} - \frac{A}{p} - \pi(z, p; \mu, \beta), \quad (14a)$$

$$\frac{\dot{z}}{z} = \frac{1}{\sigma[(p-1)/\varphi+1]} \left\{ \frac{1}{\varphi} p \left[\frac{A}{p} - (\rho + \delta) \right] + \left(1 - \frac{1}{\varphi} \right) \left[\frac{p-1}{\varphi} - (\pi + \rho) \right] \right\} - [(1-\beta)A - \delta - z]. \quad (14b)$$

Thus, the equilibrium conditions are simplified to a stationary system summarized by the above two equations and determine the equilibrium paths of p and z . The equilibrium path of π is in turn determined by (12c). The economic growth rate is determined by (7a).

A long-run equilibrium in an endogenous growth model is characterized by a *Balanced Growth Path* (henceforth, BGP), along which all perpetual growing variables change at a constant rate. In this model, a BGP is reached when $\dot{p} = 0$ and $\dot{z} = 0$. Let a value along a BGP be denoted by the value with an asterisk. We are ready to envisage the existence of a BGP. It suffices to analyze (14a) and (14b) in a (z, p) plan as follows.

First, we investigate the relationship $\dot{z} = 0$. If we substitute for $\pi(z, p)$ from $\dot{p} = 0$, the relationship $\dot{z} = 0$ becomes

$$\frac{A}{p} - \rho - \delta = \sigma[(1-\beta)A - \delta - z], \quad (15a)$$

which, for convenience, is referred to as locus CK.

Next, if we use the relationship $\dot{p} = 0$, along with the conditions that in a BGP $\dot{k}/k = \dot{m}/m$, we obtain

$$\frac{p-1}{\varphi} - \frac{A}{p} + (1-\beta)A = z + \mu, \quad (15b)$$

which is referred to as locus PP.

In Appendix 3, we have shown that both loci CK and PP are positively sloping in a (z, p) plan. Moreover, we have shown that under a mild condition, the value of p for locus PP at $z=0$ is higher than the corresponding value of p for locus CK. See Figure 4. Therefore, there exists a

unique BGP.

[Insert Figure 4 about here]

4.1 Balanced Growth Effect

We are ready to examine the effects of alternative methods of public finance along a BGP. As in the optimal growth model, we assume that initially there is no government expenditure with zero consumption and seignorage tax rates; i.e., $\beta=\tau_c=\mu=0$. If we denote the rate of economic growth by θ , then we obtain $\theta=(1-\beta)A-\delta-z$ according to (7a).

Now, suppose that the government increases public spending permanently. A balance of government budget in (5) indicates the following constraint.

$$d\beta = \frac{1}{A}(xd\mu + zd\tau_c),$$

where $x=m/k$ denotes the ratio of real balances to capital and, using (12b), is

$$x = \varphi(1-\beta)A + (1+\tau_c-\varphi)z. \quad (16)$$

First, if public spending is financed only by a consumption tax, the consumption tax rate is

$$d\tau_c = \frac{A}{z}d\beta. \quad (17a)$$

Alternatively, if a higher growth rate of monetary supply is used, the seignorage tax rate is

$$d\mu = \frac{A}{x}d\beta. \quad (17b)$$

We now examine the effects of alternative finances in a BGP.¹⁶ Under a consumption tax, both Loci CK and PP are shifted leftward, because under a given amount of household budget, the amount of consumption is reduced by consumption taxes. See Figure 4. As a result of lower consumption, the effect on the consumption to capital ratio (z) is negative. There is a neutral effect on economic growth along a BGP, but the effect on welfare along a BGP is negative because of lower consumption.

Similarly, the CK and the PP loci are shifted leftward under a seignorage tax where two effects are at work (see Figure 4). A seignorage tax raises inflation and reduces consumption and tends to decrease the ratio of consumption to capital. Moreover, a higher inflation reduces real balances that make an agent reduce investment and thus capital formation. The ratio of consumption to capital in a seignorage tax is lower than initial BGP, but is higher than that in a consumption tax as a result of a low growth rate of capital in a seignorage tax. Therefore, a seignorage tax always has a negative effect on welfare along a BGP. Thus, both ways of public finance have adverse effects

¹⁶ See Appendix 4 for details about the comparative-static analysis.

on welfare along a BGP. To compare their relative ranking, we resort to a quantitative exercise.

To calibrate our model, the same parameter values as those in the optimal growth model are used for δ , ρ , σ , β and φ . The value of A is calibrated in conformance with the 2% long-run economic growth rate in the U.S.A., and we obtain $A=0.145$. Under this set of parameter values, the equilibrium along a BGP is as follows: $z^*=0.0750$, $p^*=1.0350$ and $\theta^*=0.02$. We also obtain the discounted lifetime utility at $U^*=-446.9894$.¹⁷ We next quantify the effects of the two permanent tax policies. Note that in our endogenous growth model, a permanent shock leads the equilibrium to adjust instantaneously toward a new BGP.

Suppose that public spending is permanently increased from 0 to 10% of total output from now on.¹⁸ A permanent source of revenue is required to balance the government budget in a BGP, where either the consumption tax rate or the seignorage tax rate is increased. The required consumption or seignorage tax rates and the resulting effects on consumption, the ratio of consumption to capital and the rate of economic growth are illustrated in panel A of Figure 5.

[Insert Figure 5]

First, in the case of a permanent consumption tax, economic growth is not changed as a permanent consumption tax is neutral to capital formation. As consumption is reduced permanently, the ratio of consumption to capital is reduced permanently.

Next, in the case of a permanent seignorage tax, the effect on economic growth is negative, as opposed to a neutral effect under a permanent consumption tax. However, the negative effect on consumption is less severe than that under a permanent consumption tax. As the result of the two effects, the ratio of consumption to capital in a permanent seignorage tax is permanently higher than the corresponding ratio in a consumption tax.

As a result of a permanently lower level of consumption, the welfare in a permanent consumption tax is lower than the welfare in a seignorage tax. As in an optimal growth model, this welfare ranking order maintains as long as there is any fraction of investment that is cash constrained. See panel A, Table 2. Only in the special case when investment is not cash

¹⁷ Welfare is calculated in the following manner. Along a BGP, $k(t)=k_0e^{0t}$ and $c(t)=c_0e^{0t}$. Thus, the discounted lifetime utility may be written as $U^* = \frac{1}{(1-\sigma)} \left[\frac{z_0^{1-\sigma} k_0^{1-\sigma}}{\rho-\theta^*(1-\sigma)} - \frac{1}{\rho} \right]$. As k_0 is predetermined, we normalize $k_0=1$. Therefore, it is reasonable to obtain $z_0^*=z^*=0.0750$ in accordance with a BGP. In this way, we quantify the value of the discounted lifetime utility.

¹⁸ See Appendix 5 for technical details regarding how we determine the equilibrium paths of endogenous variables under a permanent shock.

constrained ($\phi=0$), the level of consumption is at the same level under alternative ways of taxes and thus the welfare costs are the same. However, when a fraction of investment is cash constrained, a permanent seignorage tax will then have a lower relative price of consumption (to investment) than that in a permanent consumption tax. As a result, the level of consumption in a seignorage tax is permanently higher than that in a consumption tax lower. Therefore, the welfare cost in a permanent consumption tax is higher than that in a permanent seignorage tax.

[Insert Table 2 here]

4.2 Dynamic Effects

Now, we consider a temporary shock. As the equilibrium adjusts instantaneously to a BGP in the event of a permanent shock, a temporary shock serves the purpose in the construction of an equilibrium path exhibiting transitional dynamics.

Suppose that public expenditure is increased temporarily from 0% to 10% of total output in the next 10 periods.¹⁹ In order to balance the government budget in a BGP, the temporary expenditure is financed either by a temporary consumption tax rate or a temporary seignorage tax rate. The required consumption tax rate or seignorage tax rate and the resulting effects are illustrated in panel B of Figure 5.

First, in the case of a temporary consumption tax change, consumption is temporarily reduced, with the level of consumption higher than that under a permanent consumption tax because of a consumption-smooth effect that borrows against the future. As a result, capital accumulation and thus economic growth are temporarily lower. Next, in a temporary seignorage tax, both consumption and capital accumulation are lower.

Comparing the two taxes, a temporary seignorage tax has only an indirect adverse effect on consumption due to a moderate higher relative price of consumption to investment while a consumption tax had a direct adverse effect via a strong higher relative price of consumption to investment. In early periods of policy changes the direct effect dominates, making the level of consumption lower in a consumption tax. However, a consumption tax is neutral to capital formation, while a seignorage tax is adverse to capital accumulation leading to a lower growth rate of capital in early periods of a policy change. Thus, the ratio of consumption to capital is lower in a consumption tax than in a seignorage tax in early periods of policy changes. As the economic growth rate is inversely related to the ratio of consumption to capital, the economic growth rate in a

¹⁹ See Appendix 6 for technical details regarding how we determine the equilibrium paths of endogenous variables under a temporary shock.

consumption tax is decreasing over the periods of policy changes which is eventually smaller than the economic growth rate in a seignorage tax in latter periods of policy changes. In latter periods the level of consumption is lower in a seignorage tax.

The level of consumption in a consumption tax is lower than that in a seignorage tax in the early period of a policy change, albeit it is higher in the later period of a policy change. Thus, a welfare comparison may depend on the discounting. Our numerical exercises indicate that no matter whether under the discount rate in our original calibration or any other feasible discount rates, the effect in early periods always dominates and the welfare cost of a temporary consumption tax is always higher than that of a temporary seignorage tax (panel B, Table 2).

Moreover, as in an optimal growth model, the above welfare ranking order holds true even if there is no investment that is cash constrained. It is because both temporary taxes generate an intertemporal substitution effect away from current toward future consumption while a temporary consumption tax marks up and thus twists a higher price of consumption in early periods. This makes consumption in a consumption tax lower in early periods whose adverse effect on welfare dominates the effect of higher consumption in a consumption tax than in a seignorage tax later.

Therefore, we conclude that the welfare ranking order in an endogenous growth model is not different from that in an optimal growth model.

Proposition 3 *In an endogenous growth model, when equilibrium paths along both transitions and balanced growth paths are taken into account,*

- (i) a permanent consumption tax always has a higher welfare cost than a permanent seignorage tax if investment is cash constrained;*
- (ii) a temporary consumption tax always has a higher welfare cost than a temporary seignorage tax.*

5 Concluding Remarks

Monetary economists have devoted considerable effort to measuring the welfare cost of inflation. One strand of research investigates monetary policy in a public finance approach. Most studies in this body of research found results with an optimal inflation tax higher than the Friedman rule. They mostly compared the welfare between a seignorage tax and an income tax as alternative ways of public finance. Moreover, they only compared the welfare in the long run. This paper adds to this line of literature in two perspectives. First, we compare the welfare of a seignorage tax with a consumption tax. Second, we compare the welfare by taking into account

equilibrium paths along transitional dynamics.

We start by studying an otherwise standard optimal growth model with a CIA constraint and with an exogenous stream of public spending financed by either a seignorage tax or a consumption tax. In this model, when we compare only equilibrium paths in a steady state, the welfare cost of a seignorage tax is always higher than the welfare cost of a consumption tax unless investment is not cash constrained. This represents the result of the negative relationship between money and capital in a steady state when a fraction of investment is with a cash constraint. However, when the comparisons are made to include equilibrium paths along transitional dynamics, the welfare ranking order is reversed. In permanent tax changes, we find that the welfare cost of a seignorage tax is always smaller than the welfare cost of a consumption tax when there is any fraction of investment that is cash constrained. The relative prices of consumption to investment in the two taxes are twisted by cash constraints on investment. In temporary tax changes, the same ranking order holds true even without requirement of a cash constraint on investment because temporary taxes bring in an intertemporal substitution effect away from current toward future consumption while a temporary consumption tax marks up the relative price of consumption to investment in early periods. As a robustness check, we extend the model to an endogenous growth model. We find that the same ranking order in an optimal growth model holds true in an endogenous growth model. As our results indicate, in financing public spending, an inflation tax is preferred to a consumption tax and the inflation tax is higher than the Friedman rule.

Overall, our paper may have shortcomings that can be improved upon but at the same time offers directions for further research. First, as in existing strand of research, our model is analyzed in a deterministic framework. In a stochastic framework, inflation may bring out fluctuations. Then, some may be curious to find that the welfare cost of inflation may no longer be lower than that of a consumption tax in a stochastic economy. We think that it is not possible to reverse the welfare ranking order as long as the inflation rate is at a low level. However, in the case with high inflation, speculations about the future prices may distort resource reallocation and bring about an extra cost. Then, we are not sure of the resulting effect on a welfare ranking order. It may be interesting to extend our model into a stochastic version with high inflation and study those consequences. Next, there are different ways of exercising money supply implementation policy; they may be in a policy of transfer, open market operations or other policies. If these implementation policies do not effect through a lump-sum payment, the welfare cost of a seignorage tax may be different depending upon the ways of implementation. Studies of the

welfare costs of a seignorage tax under different implementation policies with each welfare cost in comparisons with the welfare cost of a consumption tax are an avenue for further research.

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Table 1 Welfare in an optimal growth model: various discounting and cash constraints on investment

φ	ρ	Required rate of consumption tax τ_c^* (%)	Percentage change in welfare (%)	Required rate of seignorage tax μ^* (%)	Percentage change in welfare (%)
A. Permanent changes					
0.5	1.00	11.23	-99.0565	11.17	-98.1057
0.5	0.50	11.40	-78.4142	11.25	-77.4596
0.5	0.10	12.48	-52.6291	11.72	-51.8326
0.5	0.04	13.68	-27.9980	12.19	-27.5453
0.5	0.01	15.55	-25.6620	12.83	-25.1569
0.5	0.001	16.77	-27.2516	13.20	-26.6229
0.01	0.04	13.74	-27.6351	13.70	-27.6195
0.001	0.04	13.74	-27.6286	13.74	-27.6272
0	0.04	13.74	-27.6282	13.74	-27.6282
B. Temporary changes					
0.5	1.00	11.23	-99.0565	11.17	-98.1131
0.5	0.50	11.40	-78.5700	11.25	-77.6641
0.5	0.10	12.48	-28.0635	11.72	-27.8211
0.5	0.04	13.68	-4.8822	12.19	-4.8177
0.5	0.01	15.55	-0.8789	12.83	-0.8590
0.5	0.001	16.77	-0.0974	13.20	-0.0950
0	0.04	13.74	-4.7657	13.74	-4.7572

Note:

1 Baseline parameters are $\rho=0.04$ and $\varphi=0.5$. Other common parameters are $\sigma=2.5$, $A=1$, $\alpha=0.31$ and $\delta=0.05$.

2 The percentage change in welfare is the percentage change of the welfare of the change in a tax rate under $\beta=0.1$ from the welfare in the benchmark case under $\beta=0$.

3 Value τ_c^* and μ^* are the consumption tax rate and the growth rate of money in steady state.

Table 2. Welfare in an AK model: various discounting and cash constraints on investment

φ	ρ	Required rate of consumption tax τ_c (%)	Percentage change in welfare (%)	Required rate of seignorage tax μ (%)	Percentage change in welfare (%)
A. Permanent changes					
0.5	0.06	21.12	-39.8623	15.89	-38.7474
0.5	0.04	23.98	-45.0778	14.90	-43.5451
0.5	0.01	30.11	-60.2580	14.31	-57.5297
0.5	0.001	32.62	-281.5676	16.21	-267.6153
0	0.04	24.79	-46.8459	24.79	-46.8459
B. Temporary changes					
0.5	0.06	21.12	-26.0238	15.89	-25.8104
0.5	0.04	23.98	-26.4988	14.90	-26.2953
0.5	0.01	30.11	-29.2141	14.31	-29.0230
0.5	0.001	32.62	-127.5273	16.21	-126.7481
0	0.04	24.79	-27.3948	24.79	-27.3449

Note

- 1 Baseline parameters are $\rho=0.04$ and $\varphi=0.5$. Other common parameters are $\sigma=2.5$, $A=0.145$ and $\delta=0.05$.
- 2 If the time preference rate is above 0.06, then the economic growth rate is negative
- 3 Percentage change in welfare is the percentage change of the welfare of the change in a tax rate under $\beta=0.1$ from the welfare in the benchmark case under $\beta=0$.
- 4 Value τ_c^* and μ^* are the consumption tax rate and the growth rate of money in steady state.

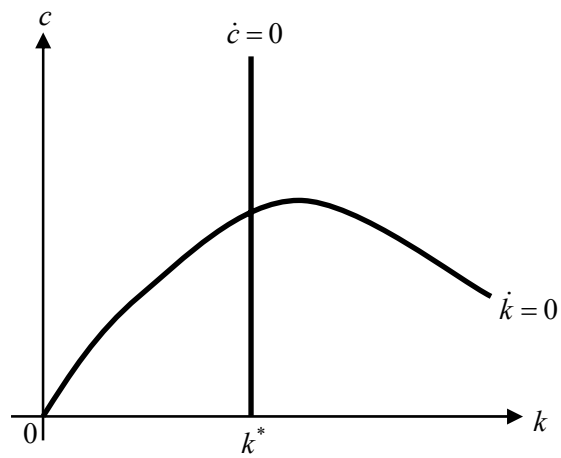


Figure 1. A steady state in an optimal growth model

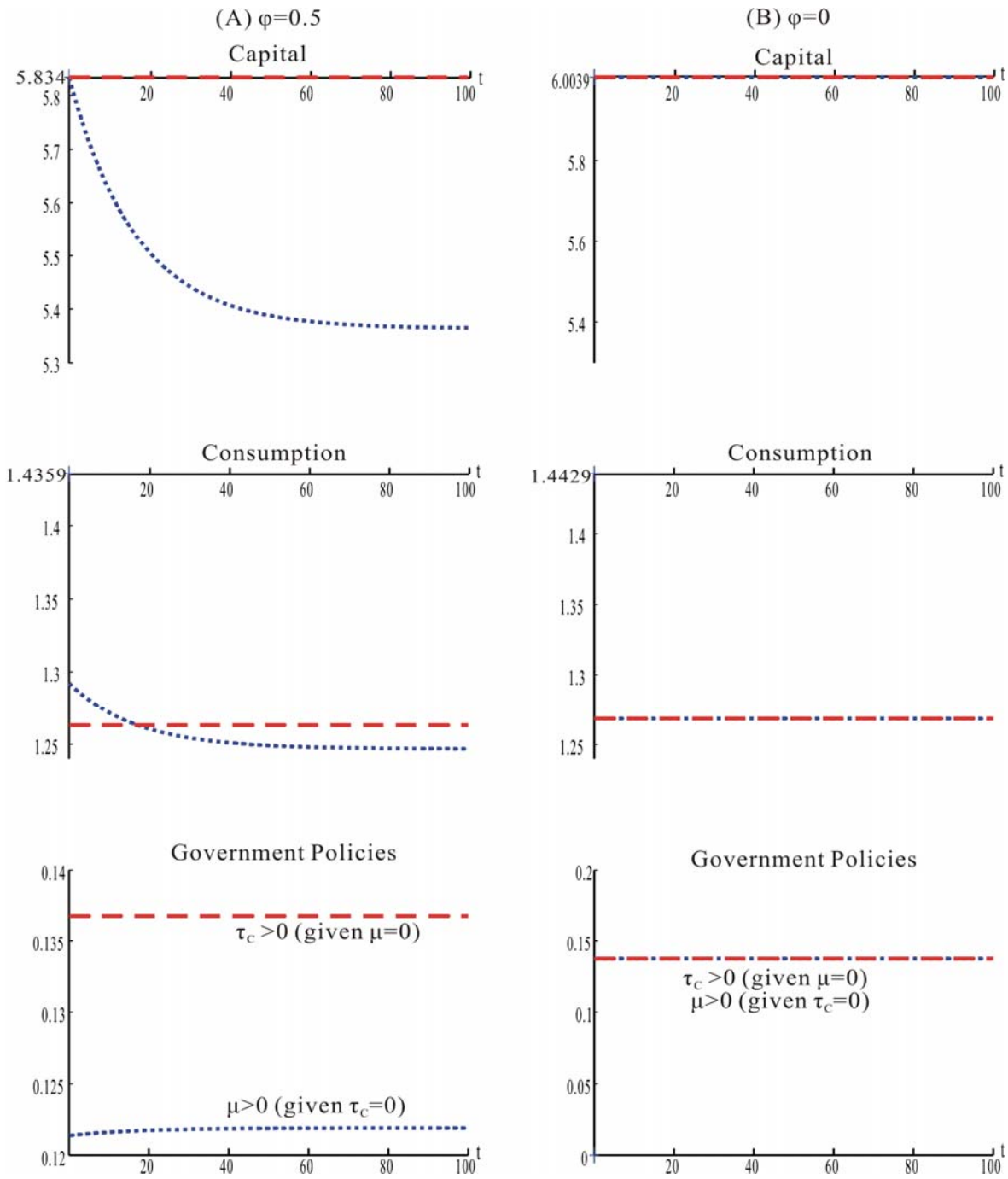


Figure 2. Effects of alternative permanent ways of public finance in optimal growth model

- Note 1. \dots : $\beta=1\%$, $\mu>0$, and $\tau_c=0$; $---$: $\beta=1\%$, $\tau_c>0$ and $\mu=0$. Other parameters are $\sigma=2.5$, $A=1$, $\alpha=0.31$, $\delta=0.05$ and $\rho=0.04$.
2. Initial steady state, under $\beta=\tau_c=\mu=0$, is represented by the intersection of the horizontal and the vertical axes (except for the graph of government policies in the case of $\phi=0.5$).

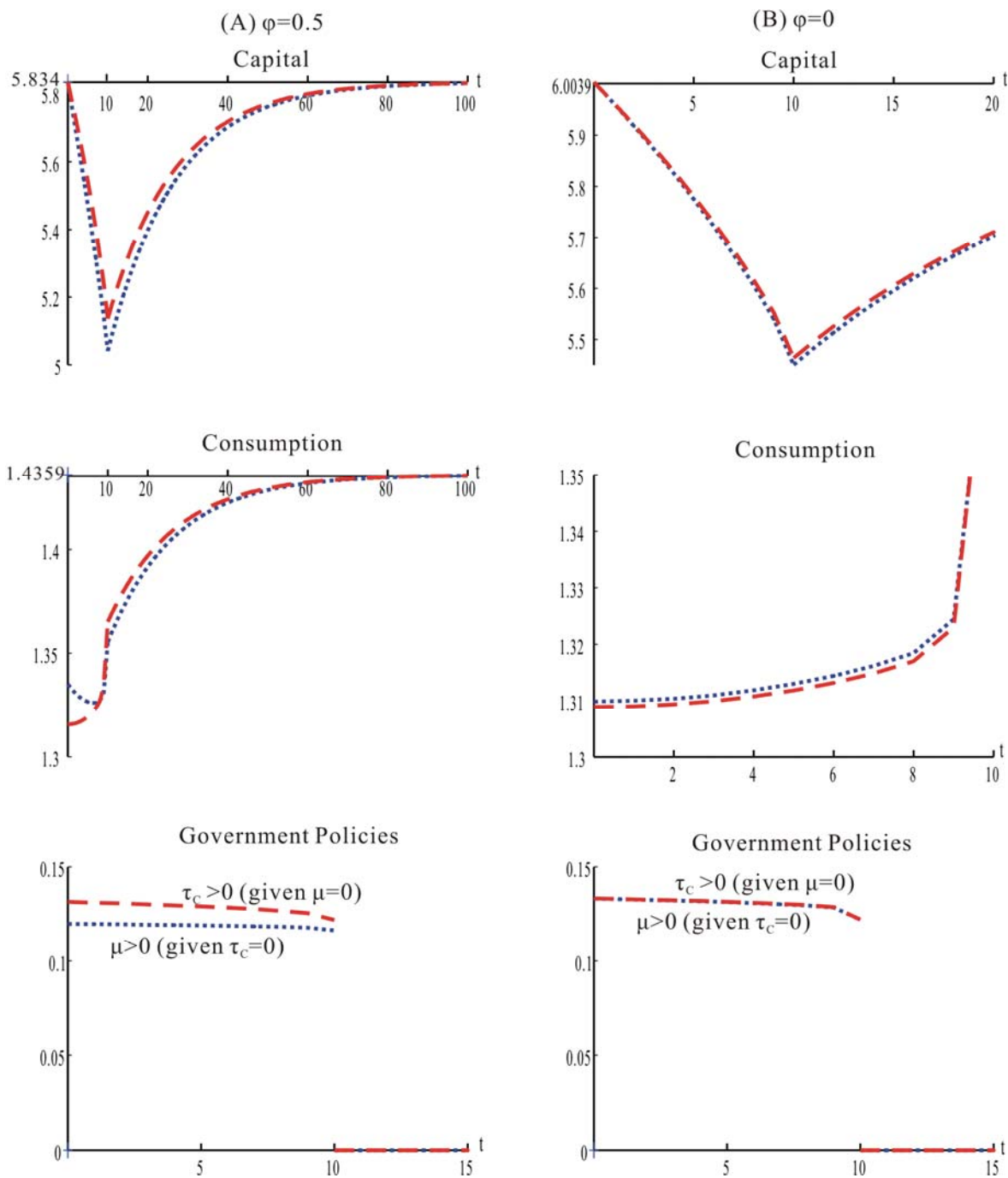


Figure 3. Effects of alternative temporary ways of public finance in an optimal growth model

Note 1. \cdots : $\beta=1\%$, $\mu>0$, and $\tau_c=0$; $---$: $\beta=1\%$, $\tau_c>0$ and $\mu=0$. Other parameters are $\sigma=2.5$, $A=1$, $\alpha=0.31$, $\delta=0.05$ and $\rho=0.04$.

2. Initial steady state, under $\beta=\tau_c=\mu=0$, is represented by the intersection of the horizontal and the vertical axes (except for the graph of consumption in the case of $\phi=0$).

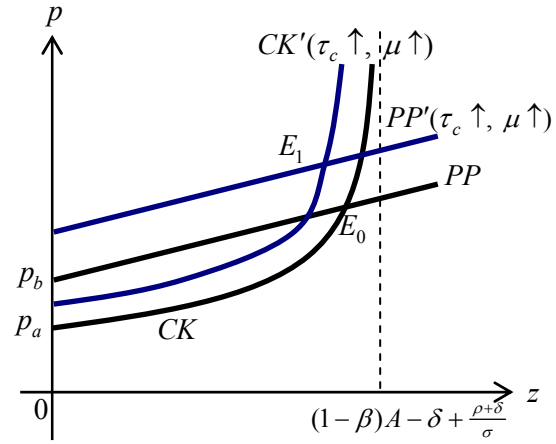


Figure 4. BGP and comparative-static effects in an endogenous growth model

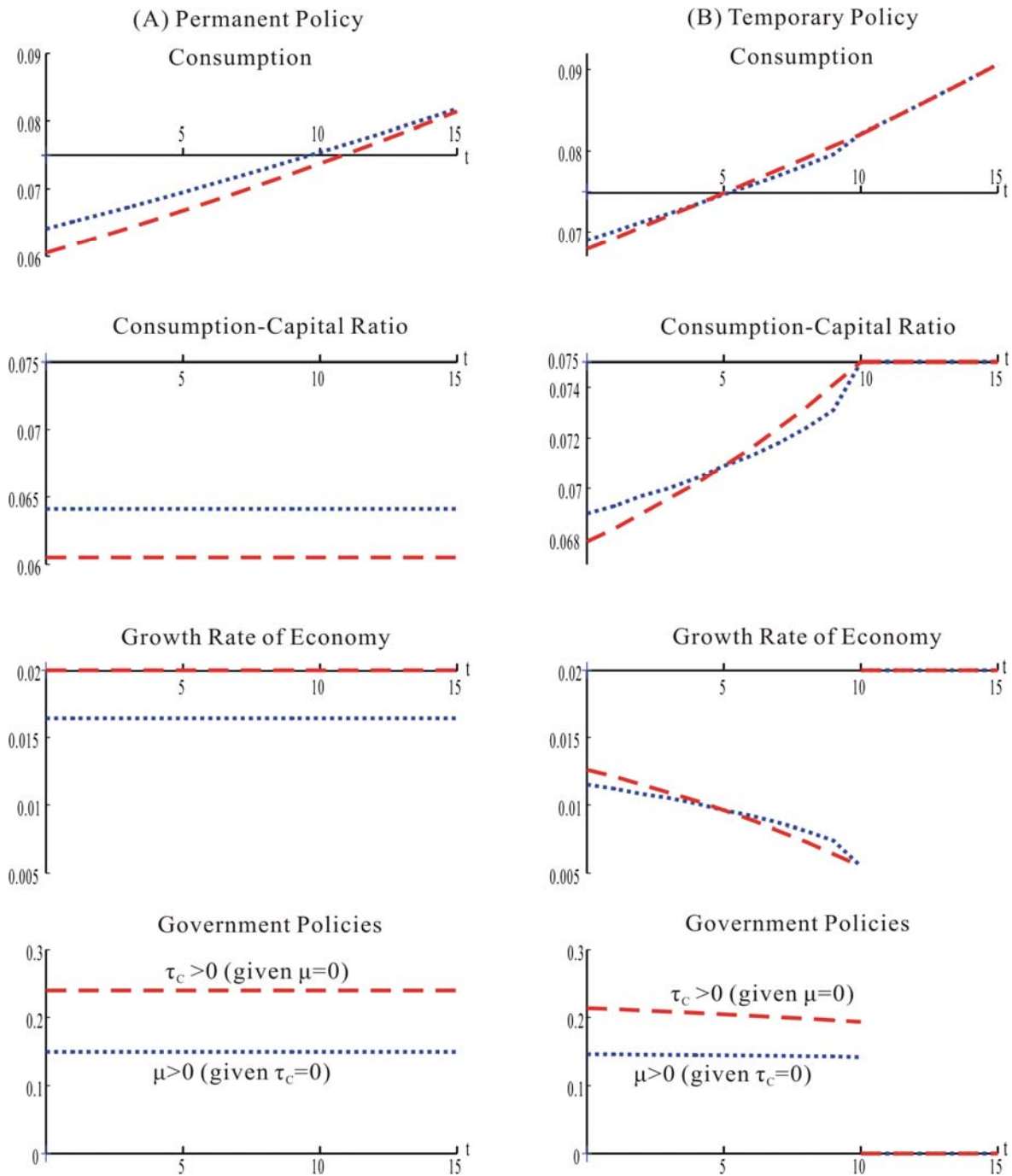


Figure 5. Effects of alternative ways of public finance in an Ak model

- Note. 1. \cdots : $\beta=1\%$, $\mu>0$, and $\tau_c=0$; $---$: $\beta=1\%$, $\tau_c>0$ and $\mu=0$. Other parameters are $\sigma=2.5$, $A=0.145$, $\delta=0.05$, $\rho=0.04$ and $\varphi=0.5$.
2. Initial steady state, under $\beta=\tau_c=\mu=0$, is represented by the intersection of the horizontal and the vertical axes.

Appendix to “The Welfare Cost of Inflation Tax and Consumption Tax”

Been-Lon Chen, Mei Hsu, and Chia-Hui Lu

Appendix 1 Dynamic Equilibrium Path of Permanent Shocks in an Optimal Growth Model

Taking a linear Taylor’s expansion of the equilibrium system (7b), (8b) and (8d) in the neighborhood of the unique steady state obtains

$$\begin{bmatrix} \dot{c} \\ \dot{k} \\ \dot{\lambda}_m \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ -1 & J_{22} & 0 \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \begin{bmatrix} c - c^* \\ k - k^* \\ \lambda_m - \lambda_m^* \end{bmatrix}, \quad (\text{A1})$$

where $J_{11} = \frac{(1+\tau_c)}{u''(c^*)} (1 - \frac{1}{\varphi}) J_{31} + \rho + \delta$,

$$J_{12} = \frac{(1+\tau_c)}{u''(c^*)} [(1 - \frac{1}{\varphi}) J_{32} - \frac{1}{\varphi} \lambda_m f''(k^*)],$$

$$J_{13} = \frac{(1+\tau_c)}{u''(c^*)} [(1 - \frac{1}{\varphi}) J_{33} + \frac{(\rho+\delta)(1-\varphi)}{\varphi} - \frac{1}{\varphi} f'(k^*)],$$

$$J_{22} = (1 - \beta) f'(k^*) - \delta,$$

$$J_{31} = \frac{\lambda_m^*}{\eta^*} [(1 + \tau_c - \varphi)(1 - \frac{1}{\varphi} - \rho - \delta) + \varphi(1 - \beta) f'(k^*)] + \frac{u''(c^*)}{(1+\tau_c)},$$

$$J_{32} = \frac{\lambda_m^*}{\eta^*} \{ (1 + \tau_c - \varphi) \frac{f''(k^*) \lambda_m^* (1+\tau_c)}{\varphi u''(c^*)} - \varphi(1 - \beta) f'(k^*) [(1 - \beta) f'(k^*) - \delta] \},$$

$$J_{33} = \rho + \mu + 1 - \frac{\lambda_m^* (1+\tau_c - \varphi)(1+\tau_c)}{\eta^* u''(c^*)} \{ (1 - \frac{1}{\varphi})(\rho + \mu + 1) + \frac{1}{\varphi} [(\rho + \delta)(1 - \varphi) - f'(k^*)] \},$$

$$\eta^* = (1 + \tau_c - \varphi) c^* + \varphi(1 - \beta) f(k^*) + (1 + \tau_c - \varphi) \frac{(1+\tau_c)}{u''(c^*)} (1 - \frac{1}{\varphi}) \lambda_m^*.$$

The equilibrium dynamic system involves one variable whose initial value is predetermined and two control variables which may adjust instantaneously. Thus, the dynamic equilibrium path toward a steady state is unique if the characteristic function in association with the Jacobean matrix in (A1) has only one negative eigenvalue.

Let s_i , $i=1, 2, 3$, be the eigenvalue of the Jacobean matrix in (A1) and s_1 be the negative eigenvalue. Then, equilibrium time paths for consumption, capital stock, and the shadow prices of real balances are as follows.

$$\begin{aligned} c_t &= c^* + v_{11} e^{s_1 t} b_1 + v_{12} e^{s_2 t} b_2 + v_{13} e^{s_3 t} b_3 \\ k_t &= k^* + v_{21} e^{s_1 t} b_1 + v_{22} e^{s_2 t} b_2 + v_{23} e^{s_3 t} b_3, \\ \lambda_{mt} &= \lambda_m^* + v_{31} e^{s_1 t} b_1 + v_{32} e^{s_2 t} b_2 + v_{33} e^{s_3 t} b_3 \end{aligned} \quad (\text{A2})$$

where $v_{ji}, j=1, 2, 3$, is the eigenvector corresponding to $s_i, i=1, 2, 3$, and coefficient $b_i, i=1, 2, 3$, is determined by boundary conditions.

To determine b_i , note that each of all the three variables must converge to their steady-state value when $t \rightarrow \infty$. As s_2 and s_3 are positive, this is possible only if $b_2=b_3=0$. As a result, when the share of public spending is increased (to $\beta=0.1$), the equilibrium time path becomes

$$c_t = c^* + v_{11}e^{s_1 t} b_1, \quad (\text{A3a})$$

$$k_t = k^* + v_{21}e^{s_1 t} b_1, \quad (\text{A3b})$$

$$\lambda_{mt} = \lambda_m^* + v_{31}e^{s_1 t} b_1. \quad (\text{A3c})$$

Let k_0^* (k^*) be the steady state before (after) a change in the share of public spending. As capital is not affected at $t=0$, thus (A3b) must satisfy the following condition

$$k_0^* = k^* + v_{21}b_1. \quad (\text{A3d})$$

The above relationship determines coefficient b_1 . If we substitute b_1 into (A3a) and (A3c) for time 0, we obtain c_0^* and λ_{m0}^* which are adjusted instantaneously to the saddle arm. Using b_1 and (A3a)-(A3c), we obtain the equilibrium time paths of all variables at any point of time.

Appendix 2 Dynamic Equilibrium Path of Temporary Shocks in an Optimal Growth Model

The way to determine the time path of key variables under a temporary increase in the fraction of government expenditure is similar to that in Appendix 1. The linear Taylor's expansion in (A1) and the equilibrium path in (A2) apply. The differences lie in the way in the determination of coefficients b_i . To determine b_i , note that each of the three variables in (A2) must converge to their steady-state value when $t \rightarrow \infty$. As s_2 , and s_3 are positive, this is possible only if $b_2=b_3=0$. As a result, after the moment when the share of public spending in total output is returned to the normal level ($\beta=0$), the equilibrium time path becomes

$$\begin{cases} c_t = c_0^* + v_{11}' e^{s_1 t} b_1' \\ k_t = k_0^* + v_{21}' e^{s_1 t} b_1' \\ \lambda_{mt} = \lambda_{m0}^* + v_{31}' e^{s_1 t} b_1' \end{cases} \quad \text{when } t \geq 10^+, \quad (\text{A4a})$$

where 10^+ indicates the moment just past period 10 when the share of public spending in total output is returned to zero.

The equilibrium time paths of key variables before time 10 are

$$\begin{cases} c_t = c^* + v_{11}e^{s_1 t}b_1 + v_{12}e^{s_2 t}b_2 + v_{13}e^{s_3 t}b_3 \\ k_t = k^* + v_{21}e^{s_1 t}b_1 + v_{22}e^{s_2 t}b_2 + v_{23}e^{s_3 t}b_3 \\ \lambda_{mt} = \lambda_m^* + v_{31}e^{s_1 t}b_1 + v_{32}e^{s_2 t}b_2 + v_{33}e^{s_3 t}b_3 \end{cases}, \text{ when } t = 0^+ \sim 10^-, \quad (\text{A4b})$$

where 0^+ indicates the moment when the share of public spending in total output is increased to 0.1, 10^- indicates the moment before the share of public spending in total output is returned to 0, c_0^* , k_0^* and λ_{m0}^* (*resp.* c^* , k^* , λ_m^*) are the steady state at $\beta=0$ (*resp.* $\beta=0.1$), and s_i is the eigenvalue of the Jacobean matrix in (A1) associated with $\beta=0.1$ while s_i' is the eigenvalue of the Jacobean matrix (A1) associated with $\beta=0$. The eigenvector for s_i is v_{ji} and the eigenvector for s_i' is v_{ji}' . The four coefficients, b_1, b_2, b_3 and b_1' , are determined as follows.

First, at $t=0$ when the share of public spending is increased, the value of capital is not affected. Thus, we obtain the following condition

$$k_0^* = k^* + v_{21}b_1 + v_{22}b_2 + v_{23}b_3. \quad (\text{A5a})$$

Next, because the representative agent's expectations are perfect foresighted, the continuity of all key variables suggests that the equilibrium time paths of key variables are equal right before and after period 10 when the share of public spending is returned to $\beta=0$.

$$\begin{aligned} c^* + v_{11}e^{10s_1}b_1 + v_{12}e^{10s_2}b_2 + v_{13}e^{10s_3}b_3 &= c_0^* + v_{11}'e^{10s_1'}b_1' \\ k^* + v_{21}e^{10s_1}b_1 + v_{22}e^{10s_2}b_2 + v_{23}e^{10s_3}b_3 &= k_0^* + v_{21}'e^{10s_1'}b_1' \\ \lambda_m^* + v_{31}e^{10s_1}b_1 + v_{32}e^{10s_2}b_2 + v_{33}e^{10s_3}b_3 &= \lambda_{m0}^* + v_{31}'e^{10s_1'}b_1' \end{aligned} \quad (\text{A5b})$$

Conditions (A5a)-(A5b) determine the values for coefficients b_1, b_2, b_3 , and b_1' . With these values of coefficients we then use (A4a)-(A4b) to obtain the time paths of endogenous variables at any point of time.

Appendix 3 Existence of a BGP in the Endogenous Growth Model

To prove the existence of a unique BGP, the CK locus along a BGP in (15a) is rewritten as

$$\frac{A}{p} - \rho - \delta = \sigma[(1-\beta)A - \delta - z]. \quad (\text{A6a})$$

Note that $(1-\beta)A - \delta \geq z$, so the consistency in (A6a) requires $A \geq p(\rho + \delta)$. Moreover, in a (z, p) plan, the equilibrium relation in (15a) starts at $p_a = A / \{\rho + \delta + \sigma[(1-\beta)A - \delta]\} > 0$ and is positively sloping with an asymptote at $z = (1-\beta)A - \delta + (\rho + \delta)/\sigma$. See Figure 4.

Next, the PP locus along a BGP in (15b) is rewritten as

$$\frac{p-1}{\phi} - \frac{A}{p} + (1-\beta)A = z + \mu. \quad (\text{A6b})$$

To guarantee a unique equilibrium path toward a BGP, we have to require a positive

determinant of the Jacobean in the linearization of the equilibrium system $\dot{p} = 0$ and $\dot{z} = 0$. See Appendix 4. Such a condition indicates, $\frac{1}{\varphi} + \frac{A}{p^2} - \pi_p > 0$, which then implies that the slope of $\dot{p} = 0$ is positive.

Moreover, when $z=0$, then $\pi = \mu - (1-\beta)A + \delta$ and locus PP intersects the p axis at

$$p_b = \frac{1}{2} \{ [1 + \varphi(\mu - (1-\beta)A)] + \Phi \} > 0 \quad \text{and} \quad p_c = \frac{1}{2} \{ [1 + \varphi(\mu - (1-\beta)A)] - \Phi \} < 0,$$

where $\Phi \equiv \{ [1 + \varphi(\mu - (1-\beta)A)]^2 + 4\varphi A \}^{1/2} > 0$. Furthermore, when z goes to ∞ , locus PP approaches to a finite value of p characterized by

$$[\delta\varphi p + (p-1)p - A\varphi] \{ (1-\varphi)(1 + \tau_c - \frac{1}{\varphi}) + \sigma(1-\varphi)[\frac{1}{\varphi}(1-p) - (1 + \tau_c)] \} = \varphi p \times \\ \left\{ \sigma[\frac{1}{\varphi}(1-p) - (1 + \tau_c)] \{ \mu - \varphi[\mu - (1-\beta)A] \} - (\varphi-1)[(1 + \tau_c - \frac{1}{\varphi})(\rho - \frac{p-1}{\varphi}) + \frac{1}{\varphi}p(\rho + \delta - \frac{A}{p})] \right\}.$$

In order to assure that the PP locus intersects the CK locus and thus there exists a BGP, it is required that $p_b > p_a$. It suffices to impose

$$\text{Condition E: } \frac{A}{\rho + \delta + \sigma[(1-\beta)A - \delta]} < 1 + \varphi \{ \mu + \rho + [(1-\beta)A - \delta](\sigma - 1) \}.$$

Apparently, if $\sigma > 1$, Condition E always holds. If $\sigma < 1$, the Condition E needs a sufficiently small A or sufficiently large ρ and δ . Then, under Condition E, there exists a BGP.

Appendix 4 Comparative Statics in an Ak Model

Before conducting comparative statics, we derive some useful properties of the inflation equation to be used below. The inflation equation in (12c) is rewritten below.

$$\pi = \frac{x(z, p; \mu, \tau_c, \beta)}{s(z, p; \tau_c, \beta)} \equiv \pi(z, p; \mu, \tau_c, \beta),$$

where $s(z, p; \tau_c, \beta) \equiv -\frac{\sigma}{\varphi} [p - 1 + \varphi] \{ (1 + \tau_c)z + \varphi[(1-\beta)A - z] \} - [(1-\varphi)^2 + \tau_c(1-\varphi)] \frac{z}{\varphi} < 0$, as

$$p > 1 \quad \text{and} \quad (1-\beta)A \geq z + \delta.$$

$$x(z, p; \mu, \tau_c, \beta) \equiv -\frac{\sigma}{\varphi} (p-1 + \varphi) \{ [(1 + \tau_c)z + \varphi((1-\beta)A - z)] \mu - \varphi(1-\beta)A[(1-\beta)A - \delta - z] \} \\ - z(1 + \tau_c - \varphi) \left[(1 - \frac{1}{\varphi})(\rho - \frac{p-1}{\varphi}) + \frac{1}{\varphi}p(\rho + \delta - \frac{A}{p}) \right].$$

It is straightforward to derive

$$\pi_z = \frac{\partial \pi}{\partial z} = \frac{-\sigma\varphi(1-\beta)A}{s(z, p; \tau_c, \beta)} \left[\frac{1}{\varphi}(p-1) + 1 \right] > 0,$$

$$\pi_p = \frac{\partial \pi}{\partial p} = \frac{z(1 + \tau_c - \varphi)}{s(z, p; \tau_c, \beta)\varphi} \left[1 - \frac{1}{\varphi} - \frac{A}{p} \right],$$

$$\pi_\mu = \frac{\partial \pi}{\partial \mu} = \frac{-\sigma}{s(z, p; \tau_c, \beta)} \left[\frac{1}{\varphi}(p-1) + 1 \right] \{ (1 + \tau_c)z + \varphi[(1-\beta)A - z] \} > 0,$$

$$\pi_{\tau_c} = \frac{\partial \pi}{\partial \tau_c} = 0,$$

$$\pi_{\beta} = \frac{\partial \pi}{\partial \beta} = \frac{-\sigma\varphi(1-\beta)A^2}{s(z,p;\tau_c,\beta)} \left[\frac{1}{\varphi}(p-1) + 1 \right] = A \frac{\partial \pi}{\partial z} > 0.$$

We are ready to conduct comparative statics. Taking Taylor's expansion of system (14a) and (14b) in the neighborhood of the unique BGP obtains

$$\begin{bmatrix} \dot{p} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} p - p^* \\ z - z^* \end{bmatrix} + \begin{bmatrix} J_{13} \\ J_{23} \end{bmatrix} (\mu - \bar{\mu}) + \begin{bmatrix} J_{14} \\ J_{24} \end{bmatrix} (\tau_c - \bar{\tau}_c) + \begin{bmatrix} J_{15} \\ J_{25} \end{bmatrix} (\beta - \bar{\beta}). \quad (\text{A7})$$

where $\bar{\mu}$, $\bar{\tau}_c$, and $\bar{\beta}$ are the initial growth rate of money, consumption tax, and a fraction of the government spending in final output, respectively, and

$$J_{11} = p^* \left(\frac{1}{\varphi} + \frac{A}{p^2} - \pi_p \right),$$

$$J_{12} = -p^* \pi_z < 0,$$

$$J_{13} = -\pi_{\mu} p^* < 0,$$

$$J_{14} = -\pi_{\tau_c} p^* = 0,$$

$$J_{15} = -\pi_{\beta} p^* < 0,$$

$$J_{21} = \frac{z^* [(1/\varphi)(A/p^*) + (1-1/\varphi)(\pi_p - 1/\varphi)]}{\sigma[(1-p^*)/\varphi - 1]},$$

$$J_{22} = z^* \left\{ \frac{(1-1/\varphi)}{\sigma[(1-p^*)/\varphi - 1]} \pi_z + 1 \right\},$$

$$J_{23} = \frac{(1-1/\varphi)\pi_{\mu}}{\sigma[(1-p^*)/\varphi - 1]} z^*,$$

$$J_{24} = 0,$$

$$J_{25} = z^* \left\{ \frac{(1-1/\varphi)\pi_{\beta}}{\sigma[(1-p^*)/\varphi - 1]} + A \right\}.$$

It is straightforward to obtain $\text{Det } J = p^* z^* \left(\frac{1}{\varphi} + \frac{A}{p^2} - \pi_p \right) - \frac{z^* A}{p^* \sigma} \pi_z$. In order to assure that there exists a unique path toward the unique BGP, it is required $\text{Det } J > 0$. Note when $\text{Det } J > 0$, we obtain $\frac{1}{\varphi} + \frac{A}{p^2} - \pi_p > 0$, which then indicates a positive slope of locus $\dot{p} = 0$ in Appendix 3.

If we evaluate (A7) at $\bar{\mu} = \bar{\tau}_c = \bar{\beta} = 0$ initially, it is straightforward to obtain the following comparative results about alternative ways of public finance.

First, the effects of a permanent consumption tax in a BGP are

$$\frac{dz}{d\tau_c} = -z < 0, \quad (\text{A8a})$$

$$\frac{d\theta}{d\tau_c} = -z - \frac{dz}{d\tau_c} = -z - (-z) = 0, \quad (\text{A8b})$$

$$\frac{dU}{d\tau_c} = \frac{c_0^{-\sigma}}{\rho - \theta(1-\sigma)} \frac{dc_0}{d\tau_c} < 0, \text{ as } \frac{dc_0}{d\tau_c} < 0. \quad (\text{A8c})$$

Next, the effects of a permanent seignorage tax in a BGP are

$$\frac{dz}{d\mu} = \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial \mu} + \frac{\partial z}{\partial \mu} = -x + \frac{\partial z}{\partial \mu} > 0, \text{ if } \frac{\partial z}{\partial \mu} > x > 0, \quad (\text{A9a})$$

$$0 < \frac{\partial z}{\partial \mu} < x,$$

$$\frac{d\theta}{d\mu} = -A \frac{d\beta}{d\mu} - \frac{dz}{d\mu} = -\frac{Az^*}{p^* \sigma \text{Det} J} \pi_\mu < 0, \quad (\text{A9b})$$

$$\frac{dU}{d\mu} = \frac{c_0^{-\sigma}}{\rho - \theta(1-\sigma)} \frac{dc_0}{d\mu} + \frac{c_0^{1-\sigma}}{[\rho - \theta(1-\sigma)]^2} \frac{d\theta}{d\mu} < 0, \text{ as } \frac{d\theta}{d\mu} < -\frac{dc_0}{d\mu} \frac{\rho - \theta(1-\sigma)}{c_0}. \quad (\text{A9c})$$

Appendix 5 Dynamic Equilibrium Path of a Permanent Shock in an Ak Model

The way to determine the time paths of all variables in an Ak model is similar to that in Appendix 1. The linearization of the two by two equilibrium system is in (A7). Note that the equilibrium dynamic system involves two control variables which may adjust instantaneously. We have found that the characteristic function in association with the Jacobean matrix evaluated at a unique BGP in (A7) has two positive eigenvalues and thus there is a unique dynamic equilibrium path toward the unique BGP.

Let s_i , $i=1, 2$, be the eigenvalue of the Jacobean matrix in (A7). Then, equilibrium time paths for the ratio of the price of capital to the price of real balance and the consumption-capital ratio are as follows

$$p_t = p^* + v_{11} e^{s_1 t} b_1 + v_{12} e^{s_2 t} b_2,$$

$$z_t = z^* + v_{21} e^{s_1 t} b_1 + v_{22} e^{s_2 t} b_2,$$

where v_{ji} , $j=1, 2$, is the eigenvector corresponding to s_i , $i=1, 2$, and coefficient b_i , $i=1, 2$, is determined by boundary conditions.

Once p_t and z_t are determined, the time paths of all variables may be derived. We normalize the original BGP before a policy change so that $k_0=1$. Under the normalization, $c_0 = z_0 k_0$, where z_0 is the ratio of consumption to capital at the original BGP. As capital is predetermined, it does not change when the growth rate of nominal money is increased at time 0^+ . Using the goods market clearing condition in (7a), we obtain $k_{t+1} - k_t = \dot{k} = k_t \theta(z_t, p_t; \mu, \tau_c, \beta)$, $t=0, 1, \dots, \infty$. Once the time series of k_t is obtained, we can obtain the time series of $c_t = k_t z_t$. The welfare or the lifetime

discounted utility is just the discounted sum of all c_t : $\frac{1}{1-\sigma} \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} (c_t^{1-\sigma} - 1)$.

To determine b_i , note that both p_t and z_t must converge to their value in a new BGP when $t \rightarrow \infty$. As s_1 , and s_2 are positive, this is possible only if $b_1=b_2=0$. As a result, the equilibrium time paths after the moment when the share of public spending in total output is increased to $\beta=0.1$ become

$$\begin{cases} p_t = p^* \\ z_t = z^* \end{cases}, \text{ when } t \geq 0^+. \quad (\text{A10})$$

Finally, both p_t and z_t adjust to their BGP values immediately when a permanent policy change occurs.

Appendix 6 Dynamic Equilibrium Path of a Temporary Shock in an Ak Model

The linear Taylor's expansion in (A7) applies in the case of a temporary change in β . To determine b_i , notice that both p_t and z_t must converge to their BGP values when $t \rightarrow \infty$. As s_1 and s_2 are positive, this is possible only if $b_1=b_2=0$. As a result, when the share of public spending in total output is returned to its original level at $\beta=0$, equilibrium time paths after the moment become

$$\begin{cases} p_t = p_0^* \\ z_t = z_0^* \end{cases}, \text{ when } t \geq 10^+. \quad (\text{A11a})$$

Equilibrium time paths before period 10 are

$$\begin{cases} p_t = p^* + v_{11}e^{s_1 t} b_1 + v_{12}e^{s_2 t} b_2 \\ z_t = z^* + v_{21}e^{s_1 t} b_1 + v_{22}e^{s_2 t} b_2 \end{cases}, \text{ when } t = 0^+ \sim 10^-. \quad (\text{A11b})$$

In (A11a) and (A11b), p_0^* and z_0^* (*resp.* p^* and z^*) are the BGP value at $\beta=0$ (*resp.* at $\beta=0.1$), and all other notations are the same as those defined in Appendix 1 and 2.

Representative agent's expectations are perfect foresighted. Thus, the continuity of p_t and z_t suggests that their equilibrium time paths are equal right before and after period 10 when the share of public spending in total output is returned to $\beta=0$.

$$p^* + v_{11}e^{10s_1} b_1 + v_{12}e^{10s_2} b_2 = p_0^*, \quad (\text{A12a})$$

$$z^* + v_{21}e^{10s_1} b_1 + v_{22}e^{10s_2} b_2 = z_0^*. \quad (\text{A12b})$$

Conditions (A12a)-(A12b) determine the values for coefficients b_1 and b_2 . With these values of coefficients, we then use (A11a) and (A11b) to obtain the time paths of p_t and z_t at any point of time. Other time-series variables are determined once p_t and z_t are determined.